XII.—*On the Differential Equation of all Parabolas.*

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Contents.

§ 1. *Introduction.*

It is my object in the present paper to give the geometrical interpretation of the differential equation of all parabolas, as promised at the end of my remarks on Monge's Differential Equation to all Conics.† I have already incidentally pointed out‡ the easiest method of deriving the differential equation of all parabolas from the integral equation of the curve, *viz.*, the parabola being given by

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

where

$$h^2 = ab,$$

we have, by solving for y,

$$by = -(hx + f) \pm \left\{2(hf - bg)x + (f^2 - bc)\right\}^{\frac{1}{2}},$$

which may be written

$$y = \mathrm{P}x + \mathrm{Q} \pm \sqrt{\mathrm{R}x + \mathrm{S}},$$

and this being on both sides operated upon by $\left(\frac{d}{dx}\right)^2$, leads to

$$\frac{d^2y}{dx^2} = \mp \frac{1}{4} \frac{\mathrm{R}^2}{(\mathrm{R}x + \mathrm{S})^{\frac{3}{2}}},$$

whence

$$\left(\frac{d^2y}{dx^2}\right)^{-\frac{2}{3}} = lx + m,$$

so that

$$\left(\frac{d}{dx}\right)^2 \left(\frac{d^2y}{dx^2}\right)^{-\frac{2}{3}} = 0,$$

which is equivalent to the developed form

$$3\frac{d^2y}{dx^2}\frac{d^4y}{dx^4} - 5\left(\frac{d^3y}{dx^3}\right)^2 = 0,$$

and this is the differential equation to be geometrically interpreted. It

* For a full analysis of this paper, see P. A. S. B. 1888, pp. 156-157.

† P. A. S. B. 1888, p. 86, footnote.

‡ J. A. S. B. 1887, vol. lvi, part ii, p. 136; P. A. S. B. 1887, pp. 185-186.

seems not wholly unnecessary to point out that what we are required to do is simply the discovery of a property of the parabola, leading to a geometrical quantity which, while adequately represented by the above differential expression, vanishes at every point of every parabola. As the interpretation I propose to give, follows directly from the properties of the osculating conic of any curve, I will begin with a brief account of Transon's Theory of Aberrancy as expounded in his original memoir.*

§ 2. *Transon's Theory of Aberrancy.*

Consider the conic of closest contact at any point P of a given curve ; if NP be the normal to the conic at P, and O its centre, the line OP is called the axis of aberrancy, the point O the centre of aberrancy, and the angle NOP the angle of aberrancy, *viz*, this is the angle which measures the deviation of the curve from the circular form. Again, from the closely analogous case of the circle of curvature, we may borrow a very useful term and call the length OP, which joins P with the centre of aberrancy, the *radius of aberrancy ;* and the reciprocal of this radius may conveniently be termed the *index of aberrancy.*† Similarly, the locus of the centre of aberrancy as P travels along the given curve, may not be inappropriately termed the *aberrancy curve.* Before proceeding to obtain analytical expressions for these geometrical quantities in connection with the osculating conic, we shall first prove the following lemma :

If δ be the angle between the central diameter and the normal at any point of a conic, ρ the radius of curvature, ρ' the radius of curvature at the corresponding point of the evolute, we have

$$\tan\delta = \frac{1}{3}\frac{\rho'}{\rho}.$$

Let C be the centre of the conic, and P the given point on the perimeter ; p the perpendicular from the centre on the tangent at P ; r the central radius vector CP ; n the normal PN as limited by the axis major ; ω the angle which the normal PN makes with the axis major, and δ the angle CPN. Then, we have the well-known relations

$$p = r\cos\delta$$
$$p^2 = a^2\cos^2\omega + b^2\sin^2\omega = a^2(1 - e^2\sin^2\omega)$$

* *Recherches sur la Courbure des Lignes et des Surfaces, Journal de Mathematiques, (Liouville)* Ier Ser., t. VI (1841), pp. 191-208. For a very short notice of the subject by Prof. Cayley, see Salmon's *Higher Plane Curves*, p. 368 (Ed. 1879).

† In the case of the circle of curvature, the very expressive phrase "index of curvature," which is the reciprocal of the radius of curvature, has been now abridged into the single short term "curvature;" but whether anything has been gained by the change is doubtful.

$$n = \frac{b^2}{a} \frac{1}{\sqrt{1 - e^2 \sin^2 \omega}}.$$

Hence

$$r = \frac{p}{\cos \delta} = \frac{a\sqrt{1 - e^2 \sin^2 \omega}}{\cos \delta},$$

and

$$\frac{\sin (\omega - \delta)}{\sin \omega} = \frac{n}{r} = \frac{b^2}{a^2} \frac{\cos \delta}{1 - e^2 \sin^2 \omega},$$

whence

$$\tan \delta = \frac{e^2 \sin \omega \cos \omega}{1 - e^2 \sin^2 \omega}.$$

Now, it is well-known that the element of arc of the ellipse is given by

$$ds = \frac{b^2}{a} \frac{d\omega}{(1 - e^2 \sin^2 \omega)^{\frac{3}{2}}},$$

whence

$$\rho = \frac{ds}{d\omega} = \frac{b^2}{a} . \frac{1}{(1 - e^2 \sin^2 \omega)^{\frac{3}{2}}},$$

$$\rho' = \frac{d\rho}{d\omega} = \frac{3b^2}{a} \frac{e^2 \sin \omega \cos \omega}{(1 - e^2 \sin^2 \omega)^{\frac{5}{2}}},$$

which give

$$\frac{\rho'}{\rho} = \frac{3e^2 \sin \omega \cos \omega}{1 - e^2 \sin^2 \omega}.$$

Hence, finally,

$$\tan \delta = \frac{1}{3} \frac{\rho'}{\rho},$$

and thus the formula is seen to be true for a central conic. To establish the property for a parabola, we notice that the centre being now at infinity, the angle at any point P between the normal and the central radius vector is the angle between the normal and the diameter, which is equal to the angle which the normal makes with the principal axis; hence, we have

$$\delta = \omega.$$

But the intrinsic equation of the parabola is well-known to be given by

$$\frac{ds}{d\omega} = \frac{2a}{\cos^3 \omega},$$

where $4a$ is the latus-rectum. Hence,

$$\rho = \frac{2a}{\cos^3 \omega}$$

$$\rho' = \frac{d\rho}{d\omega} = \frac{6a \sin \omega}{\cos^4 \omega},$$

so that

$$\frac{\rho'}{\rho} = 3 \tan \omega,$$

which gives the required formula

$$\tan \delta = \frac{1}{3} \frac{\rho'}{\rho}.$$

The above formula in the case of a central conic follows also from the properties of conjugate diameters, *viz.*, if r_1 be the semi-diameter conjugate to r, we have

$$r^2 + r_1^2 = a^2 + b^2$$
$$pr_1 = ab$$
$$\rho = \frac{r_1^3}{ab}.$$

Hence

$$rdr + r_1 dr_1 = 0$$

and

$$\frac{d\rho}{ds} = \frac{3r_1^2}{ab} \frac{dr_1}{ds} = -\frac{3r_1}{ab} \frac{rdr}{ds}$$
$$= -\frac{3r}{p} \frac{dr}{ds} = 3 \tan \delta,$$

since

$$\frac{dr}{ds} = -\sin \delta, \quad \frac{p}{r} = \cos \delta.$$

Therefore

$$\tan \delta = \frac{1}{3} \frac{d\rho}{ds} = \frac{1}{3} \frac{\rho'}{\rho},$$

as before.

We now proceed to express the elements of the osculating conic in terms of the differential co-efficients. For this purpose, we remark that

$$\rho = \frac{\left\{ 1 + \left(\frac{dy}{dx}\right)^2 \right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \frac{\left(\frac{ds}{dx}\right)^3}{\frac{d^2y}{dx^2}}$$

reduces the equation

$$\rho = \frac{ds}{d\omega} = \frac{ds}{dx} \frac{dx}{d\omega}$$

to

$$\frac{d\omega}{dx} = \frac{\frac{d^2y}{dx^2}}{\left(\frac{ds}{dx}\right)^2} = \frac{\frac{d^2y}{dx^2}}{1 + \left(\frac{dy}{dx}\right)^2},$$

and we have also

$$\frac{d\rho}{dx} = \frac{\left\{1+\left(\frac{dx}{dx}\right)^2\right\}^{\frac{1}{2}}}{\left(\frac{d^2y}{dx^2}\right)^2} \left\{3\frac{dy}{dx}\left(\frac{d^2y}{dx^2}\right)^2 - \left[1+\left(\frac{dy}{dx}\right)^2\right]\frac{d^3y}{dx^3}\right\}.$$

Hence, we get

$$\tan\delta = \frac{1}{3}\frac{\rho'}{\rho} = \frac{1}{3\rho}\frac{d\rho}{d\omega}$$

$$= \frac{1}{3\rho}\frac{\frac{d\rho}{dx}}{\frac{d\omega}{dx}}$$

$$= \frac{dy}{dx} - \frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}\frac{d^3y}{dx^3}}{3\left(\frac{d^2y}{dx^2}\right)^2}.$$

Using p, q, r to denote the first, second and third differential co-efficients of y with respect to x, we have the formula for the angle of aberrancy in the now familiar form

$$\tan\delta = p - \frac{(1+p^2)\,r}{3q^2}.$$

It is easy to verify this formula when the equation of the conic is given in form

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

for the coordinates of any point being $a\cos\phi$, $b\sin\phi$, the equation of the central radius vector is

$$ay\cos\phi = bx\sin\phi,$$

and the normal is

$$\frac{ax}{\cos\phi} - \frac{by}{\sin\phi} = a^2 - b^2,$$

so that the angle between these two lines is given by

$$\tan\delta = \frac{a^2-b^2}{ab}\sin\phi\cos\phi.$$

Again, from the equation of the curve we have

$$p = -\frac{b}{a}\frac{x}{\sqrt{a^2-x^2}} = -\frac{b}{a}\cot\phi.$$

$$q = -\frac{ab}{(a^2-x^2)^{\frac{3}{2}}} = -\frac{b}{a^2\sin^3\phi}$$

$$r = -\frac{3abx}{(a^2-x^2)^{\frac{5}{2}}} = -\frac{3b\cos\phi}{a^3\sin^5\phi},$$

which give

$$\frac{r}{3q^2} = = \frac{a \sin \phi \cos \phi}{b},$$

$$1 + p^2 = \frac{a^2 \sin^2\phi + b^2 \cos^2\phi}{a^2 \sin^2\phi}$$

$$p - \frac{(1+p^2)r}{3q^2} = \frac{a^2 - b^2}{ab} \sin \phi \cos \phi$$

so that

$$\tan \delta = p - \frac{(1+p^2)r}{3q^2},$$

which is the formula to be verified.

Next, to calculate the radius of aberrancy R, let $d\omega$ the angle between two consecutive normals, and $d\psi$ the angle between two consecutive axes of aberrancy ; then, we have clearly

$$d\omega = d\psi + d\delta.$$

Again, consider the triangle formed by two consecutive radii of aberrancy and the element of arc of the given curve ; then, we have

$$\frac{\mathrm{R}}{\sin\left(\frac{\pi}{2} - \delta\right)} = \frac{ds}{d\psi}.$$

And, similarly, from the triangle formed by two consecutive normals and the element of arc of the given curve, we get

$$ds = \rho d\omega,$$

whence

$$\mathrm{R} = \rho \cos \delta \,.\, \frac{d\omega}{d\psi}.$$

But from the equation

$$\tan \delta = \frac{1}{3\rho}\, \frac{d\rho}{d\omega},$$

we have

$$\sec^2 \delta \,.\, \frac{d\delta}{d\omega} = \frac{1}{3}.\frac{\rho \frac{d^2\rho}{d\omega^2} - \left(\frac{d\rho}{d\omega}\right)^2}{\rho^2},$$

or substituting for δ, we get

$$\frac{d\delta}{d\omega} = 3\,.\,\frac{\rho \frac{d^2\rho}{d\omega^2} - \left(\frac{d\rho}{d\omega}\right)^2}{9\rho^2 + \left(\frac{d\rho}{d\omega}\right)^2}.$$

Hence

$$\frac{d\psi}{d\omega} = 1 - \frac{d\delta}{d\omega}$$

$$= \frac{9\rho^2 + 4\left(\frac{d\rho}{d\omega}\right)^2 - 3\rho\,\frac{d^2\rho}{d\omega^2}}{9\rho^2 + \left(\frac{d\rho}{d\omega}\right)^2}.$$

Therefore, from

$$R = \rho \cos \delta \, . \, \frac{d\omega}{d\psi},$$

we have easily the relation

$$R = \frac{3\,\rho^2 \left\{ 9\,\rho^2 + \left(\frac{d\rho}{d\omega}\right)^2 \right\}^{\frac{1}{2}}}{9\,\rho^2 + 4\left(\frac{d\rho}{d\omega}\right)^2 - 3\rho\,\frac{d^2\rho}{d\omega^2}}$$

We can now, without much difficulty, change the variables, and thus obtain an expression for R in terms of x and y. Thus, as we have already seen

$$\rho = \frac{(1 + p^2)^{\frac{3}{2}}}{q}$$

$$\frac{d\rho}{dx} = \frac{(1 + p^2)^{\frac{1}{2}}}{q^2} \left\{ 3\,pq^2 - r\,(1 + p^2) \right\}$$

$$\frac{d\omega}{dx} = \frac{q}{1 + p^2},$$

whence

$$\frac{d\rho}{d\omega} = \frac{(1 + p^2)^{\frac{3}{2}}}{q^3} \left\{ 3\,pq^2 - r\,(1 + p^2) \right\}.$$

Hence, we have

$$\frac{d^2\rho}{dx^2} = \frac{(1+p^2)^{\frac{1}{2}}}{q^4} \left\{ (1 + p^2) \left[q^2\,(3q^2 - 5pr) + (1 + p^2)\,(3r^2 - qs) \right] + 3\,pq^2 \left[3pq^2 - r\,(1 + p^2) \right] \right\}$$

and

$$\frac{d^2\rho}{d\omega^2} = \frac{d}{d\omega}\left(\frac{d\rho}{d\omega}\right) = \frac{dx}{d\omega}\,\frac{d}{dx}\left(\frac{d\rho}{d\omega}\right)$$

$$= \frac{(1 + p^2)^{\frac{3}{2}}}{q^5} \left\{ (1 + p^2) \left[3q^4 - 8pq^2r + (1 + p^2)\,(3r^2 - qs) \right] + 9p^2q^4 \right\}.$$

Hence, by actual calculation, we find that

$$9\rho^2 + \left(\frac{d\rho}{d\omega}\right)^2 = \frac{(1 + p^2)^4}{q^6} \left\{ r^2\,(1 + p^2) - 6pq^2r + 9q^4 \right\}$$

$$9\rho^2 + 4\left(\frac{d\rho}{d\omega}\right)^2 - 3\rho\,\frac{d^2\rho}{d\omega^2} = \frac{(1 + p^2)^5}{q^6}\,(3qs - 5r^2).$$

Therefore, finally, we get

$$R^2 = \frac{9q^2 \left\{ r^2 + (rp - 3q^2)^2 \right\}}{(3qs - 5r^2)^2}.$$

Hence, it is evident that if I be the index of aberrancy, that is to say, the reciprocal of the radius of aberrancy, we have

$$I = \frac{3qs - 5r^2}{3q \left\{ r^2 + (rp - 3q^2)^2 \right\}^{\frac{1}{2}}}.$$

It is hardly neeessary to point out that, as these formulæ hold when the origin is anywhere, they are true when the origin is taken to be the given point on the curve whose osculating conic we are considering.

If we take the tangent and normal at the given point as the axes of x and y respectively, we may easily obtain expressions for the coordinates of the centre of aberrancy, *viz.*, we have

$$X = R \sin \delta, \; Y = R \cos \delta,$$

and from the relation

$$\tan \delta = p - \frac{(1 + p^2)\, r}{3q^2},$$

we get

$$\sin \delta = \frac{3pq^2 - r\,(1 + p^2)}{\sqrt{1 + p^2} \left\{ r^2 + (rp - 3q^2)^2 \right\}^{\frac{1}{2}}}$$

$$\cos \delta = \frac{3q^2}{\sqrt{1 + p^2} \left\{ r^2 + (rp - 3q^2)^2 \right\}^{\frac{1}{2}}}.$$

Hence, the coordinate axes being the tangent and normal at any point of a given curve, the values of the coordinates of the centre of aberrancy at that point are given by

$$X = \frac{3q \left\{ 3pq^2 - r\,(1 + p^2) \right\}}{\sqrt{1 + p^2}\,(3qs - 5r^2)}$$

$$Y = \frac{9q^3}{\sqrt{1 + p^2}\,(3qs - 5r^2)}.$$

If the coordinate axes, instead of being the tangent and normal at the given point, are such that the axis of x makes an angle θ with the tangent, we have

$$\tan \theta = - \frac{dy}{dx} = - p,$$

$$\sin \theta = \frac{-p}{\sqrt{1 + p^2}}, \quad \cos \theta = \frac{1}{\sqrt{1 + p^2}},$$

and the new coordinates of the centre of aberrancy are given by the two expressions

$$X \cos\theta + Y \sin\theta = \frac{-3qr}{3qs - 5r^2}$$

$$-X \sin\theta + Y \cos\theta = \frac{-3q(pr - 3q^2)}{3qs - 5r^2}.$$

We, therefore, finally infer that if a curve be referred to rectangular axes drawn through any origin, the co-ordinates (α, β) of the centre of aberrancy at any given point (x, y) of the curve, are given in the most general form by the system

$$\alpha = x - \frac{3qr}{3qs - 5r^2}$$

$$\beta = y - \frac{3q(pr - 3q^2)}{3qs - 5r^2} *.$$

The equation of the axis of aberrancy, in its most general form, may now be at once written down, *viz.*, x, y being the coordinates of the point on the curve through which the axis of aberrancy passes, and X, Y, the current coordinates, we have for the required equation

$$\frac{X - x}{Y - y} = \frac{x - \alpha}{y - \beta} = \frac{r}{pr - 3q^2}.$$

It may usefully be noted that the values of α, β obtained above, lead to some interesting results, *viz.*, we have

$$\frac{d\alpha}{dx} = \frac{r(9q^2t - 45qrs + 40r^3)}{(3qs - 5r^2)^2},$$

$$\frac{d\beta}{dx} = \frac{(pr - 3q^2)(9q^2t - 45qrs + 40r^3)}{(3qs - 5r^2)^2},$$

so that we may put

$$\frac{d\alpha}{dx} = \lambda T,$$

$$\frac{d\beta}{dx} = \mu T,$$

where

$$\lambda = \frac{r}{(3qs - 5r^2)^2}, \quad \mu = \frac{pr - 3q^2}{(3qs - 5r^2)^2},$$

$$T \equiv 9q^2t - 45qrs + 40r^3,$$

so that

$$T = 0$$

is Monge's differential equation to all conics.† It is clear from these two expressions that if the given curve is a conic, we have

* Cf. Dublin Examination Papers, 1876, p. 152, Ques. 6, by Prof. M. Roberts.

† Cf. Dublin Examination Papers, 1880, p. 361, Ques. 5, by Prof. M. Roberts.

$$T = 0,$$

which shews that α and β are both independent of x, as is, indeed, geometrically evident, since the osculating conic of a given conic being the curve itself, the centre of aberrancy is a fixed point, *viz.*, the centre of the given conic. Similarly, if

$$\lambda = \infty\ ,\ \mu = \infty\ ,$$

we must have

$$3qs - 5r^2 = 0,$$

which shews that the given curve is a parabola, and, then the centre of aberrancy has its coordinates infinite, *viz.*, the centre of aberrancy is the centre of the parabola which is, of course, at infinity. We may also easily find the values of

$$\frac{d\alpha}{dy},\ \frac{d\beta}{dy},$$

viz., we have

$$\frac{d\alpha}{dy} = \frac{d\alpha}{dx}\frac{dx}{dy} = \frac{1}{p}\frac{d\alpha}{dx} = \lambda_1 T,$$
$$\frac{d\beta}{dy} = \frac{d\beta}{dx}\frac{dx}{dy} = \frac{1}{p}\frac{d\beta}{dx} = \mu_1 T,$$

where

$$\lambda_1 = \frac{\lambda}{p} = \frac{r}{p\,(3qs - 5r^2)^2},$$
$$\mu_1 = \frac{\mu}{p} = \frac{pr - 3q^2}{p\,(3qs - 5r^2)^2},$$

and, these results shew that when, as before,

$$T = 0,$$

the centre of aberrancy is independent of y, and, when

$$\lambda_1 = \infty\ ,\ \mu_1 = \infty\ ,$$

it is at infinity.

The directions of the principal axes of the osculating conic are also easily determined, for the conic being

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

if θ be the angle of inclination of the axis major to the axis of x, we have

$$\tan 2\theta = \frac{2h}{a - b}.$$

But, I have elsewhere* calculated the values of the constants on the right hand side in terms of the differential co-efficients, *viz.*, we have

$$\frac{h}{b} = c_3\quad ,\quad \frac{a}{b} = c_3^2 - \frac{c_1}{c_2^2},$$

where

* P. A. S. B. 1888, pp. 82—83.

$$c_1 = -\frac{U}{9q^{\frac{8}{3}}}, \quad c_2 = \frac{V}{9q^{\frac{10}{3}}},$$

$$c_3 = \frac{W}{V},$$

$$U = 3qs - 5r^2, \quad V = 3qs - 4r^2,$$

$$W = 3q^2r - pV.$$

$$T_0 = 9q^4 - 6pq^2r + (p^2 + 1) V$$

Hence, substituting, we get

$$\tan 2\theta = \frac{2c_2^2 c_3}{c_2^2 c_3^2 - c_1 - c_2^2}$$

$$= \frac{2 V W}{W^2 + 9q^4 U - V^2}$$

$$= \frac{2 (3q^2r - pV)}{9q^4 - 6pq^2r + (p^2 - 1) V} *$$

$$= \frac{2W}{T_0 - 2V}.$$

The lengths of the axes of the conic of closest contact may also be easily calculated, *viz.*, the conic being

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

and σ the length of either axis, we have the well-known equation

$$\sigma^4 + \frac{\Delta (a + b)}{(h^2 - ab)^2} \sigma^2 - \frac{\Delta^2}{(h^2 - ab)^3} = 0$$

where Δ is the discriminant. Now I have already† shewn that

$$\Delta = \frac{(h^2 - ab)^{\frac{3}{2}}}{c_1^{\frac{3}{2}}}.$$

Therefore, we have

$$\frac{\Delta (a + b)}{(h^2 - ab)^2} = \frac{a + b}{c_1^{\frac{3}{2}} (h^2 - ab)^{\frac{1}{2}}} = \frac{\frac{a}{b} + 1}{c_1^{\frac{3}{2}} \left(\frac{h^2}{b^2} - \frac{a}{b}\right)^{\frac{1}{2}}}$$

$$= \frac{c_3^2 - \frac{c_1}{c_2^2} + 1}{c_1^{\frac{3}{2}} \sqrt{\frac{c_1}{c_2^2}}} = \frac{c_2^2 (1 + c_3^2) - c_1}{c_2 c_1^2}$$

$$= \frac{9q^2 T_0}{U^2}. \ddagger$$

Similarly

* Cf. Dublin Examination Papers, 1876, p. 152, Ques. 5, by Prof. M. Roberts.

† P. A. S. B. 1888, p. 80. ‡ P. A. S. B. 1888, p. 83.

$$\frac{\Delta^2}{(h^2 - ab)^3} = \frac{1}{c_1{}^3} = -\frac{729q^8}{U^3}.$$

Therefore, the equation for the lengths of the axes reduces to

$$\sigma^4 + \frac{9q^2\,T_0}{U^2}\sigma^2 + \frac{729q^8}{U^3} = 0$$

where $T_0 = 0$ is the differential equation of all equilateral hyperbolas, and $U = 0$ of all parabolas.

If the roots of this equation be $\sigma_1{}^2$, $\sigma_2{}^2$, the area of the conic is

$$\pi\,\sigma_1\,\sigma_2 = \frac{27\pi q^4}{U^{\frac{3}{2}}},$$

a result I have obtained before.*

We may similarly consider the osculating parabola and the osculating equilateral hyperbola at any point (x, y) of a given curve. Thus, if

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

where

$$h^2 = ab$$

be the osculating parabola, and m its principal parameter, we can easily calculate m in terms of the differential coefficients from the formula

$$\frac{m}{2} = \frac{f\sqrt{a} - g\sqrt{b}}{(a + b)^{\frac{3}{2}}}.$$

For, solving for y, we have

$$y = Px + Q + \sqrt{2Hx + B}$$

where

$$P = -\frac{h}{b}, \quad Q = -\frac{f}{b}$$

$$H = \frac{hf - bg}{b^2}, \quad B = \frac{f^2 - bc}{b^2}.$$

Hence, as usual,

$$p = P + \frac{H}{(2Hx + B)^{\frac{1}{2}}}$$

$$q = \frac{-H^2}{(2Hx + B)^{\frac{3}{2}}}$$

$$r = \frac{3H^3}{(2Hx + B)^{\frac{5}{2}}},$$

so that

$$pr - 3q^2 = \frac{3PH^3}{(2Hx + B)^{\frac{5}{2}}},$$

and

* P. A. S. B. 1888, p. 84.

$$r^2 + (pr - 3q^2)^2 = \frac{9H^6(1 + P^2)}{(2Hx + B)^5},$$

whence

$$\frac{q^5}{\left\{r^2 + (pr - 3q^2)^2\right\}^{\frac{3}{2}}} = \frac{-H}{27(1 + P^2)^{\frac{3}{2}}}.$$

But since

$$H = \frac{hf - bg}{b^2} = \frac{f\sqrt{a} - g\sqrt{b}}{b^{\frac{3}{2}}}$$

$$P = -\frac{h}{b} = -\frac{\sqrt{a}}{\sqrt{b}}$$

we have from

$$\frac{m}{2} = \frac{f\sqrt{a} - g\sqrt{b}}{(a + b)^{\frac{3}{2}}},$$

the relation

$$m = \frac{-2H}{(1 + P^2)^{\frac{3}{2}}}$$

and, therefore

$$m = \frac{54q^5}{\left\{r^2 + (pr - 3q^2)^2\right\}^{\frac{3}{2}}}$$

which is accordingly the formula sought.

Again, let us investigate the coordinates of the centre of an equilateral hyperbola osculating a curve at a given point. In the first place, we know that in an equilateral hyperbola the projection of the radius of curvature at any point on the central radius vector, is equal to that radius vector; for, if R be the radius vector, δ the angle between the normal and the radius vector, ρ the radius of curvature, and a the semi-axis-transverse, we can easily show that

$$\rho = -\frac{R^3}{a^2}, \quad \cos\delta = \frac{a^2}{R^2},$$

whence

$$R = -\rho\cos\delta.$$

Hence, if an equilateral hyperbola osculates a curve at a given point, in the first instance take the tangent and normal at that point as the axes of x and y respectively; then, expressions for the coordinates of the centre are easily obtained, *viz.*,

$$X = R\sin\delta, \quad Y = R\cos\delta,$$

where R is the distance of the centre from the origin, and δ the angle between tho central radius vector and normal, so that

$$\frac{R}{\cos\delta} = -\rho = \frac{(1+p^2)^{\frac{3}{2}}}{q}.$$

But the equilateral hyperbola being a conic, we have from the preceding investigation

$$\tan\delta = p - \frac{(1+p^2)r}{3q^2},$$

whence

$$\sin\delta = \frac{3pq^2 - r(1+p^2)}{\sqrt{1+p^2}\left\{r^2 + (rp-3q^2)^2\right\}^{\frac{1}{2}}}$$

$$\cos\delta = \frac{3q^2}{\sqrt{1+p^2}\left\{r^2 + (rp-3q^2)^2\right\}^{\frac{1}{2}}}.$$

Therefore we see that the distance of the centre of the osculating equilateral hyperbola from the given point (which is the origin) is furnished by

$$R = \frac{-3q(1+p^2)}{\left\{r^2 + (rp-3q^2)^2\right\}^{\frac{1}{2}}}.$$

Hence, the coordinate axes being the tangent and normal at any point of a given curve, the values of the coordinates of the centre of the osculating equilateral hyperbola at that point are given by

$$X = \frac{3q\sqrt{1+p^2}\left\{r(1+p^2) - 3pq^2\right\}}{r^2 + (rp-3q^2)^2}$$

$$Y = \frac{3pqr\sqrt{1+p^2}}{r^2 + (rp-3q^2)^2}.$$

If the coordinate axes, instead of being the tangent and normal at the given point, are such that the axis of x makes the angle θ with the tangent, we have

$$\tan\theta = -\frac{dy}{dx} = -p$$

$$\sin\theta \frac{-p}{\sqrt{1+p^2}}, \quad \cos\theta = \frac{1}{\sqrt{1+p^2}},$$

and the new coordinates of the centre of the osculating equilateral hyperbola are given by the two expressions

$$X\cos\theta + Y\sin\theta = \frac{3qr(1+p^2)}{r^2 + (rp-3q^2)^2}$$

$$-X\sin\theta + Y\cos\theta = \frac{3q(1+p^2)(pr-3q^2)}{r^2 + (rp-3q^2)^2}.$$

We, therefore, finally infer that if a curve be referred to rectangular axes drawn through any origin, the coordinates (ξ, η) of the centre of the osculating equilateral hyperbola at any given point (x, y) of the curve, are given in the most general form by the system

$$\xi = x + \frac{3qr\,(1 + p^2)}{r^2 + (rp - 3q^2)^2}$$

$$\eta = y + \frac{3q\,(1 + p^2)\,(pr - 3q^2)^2}{r^2 + (rp - 3q^2)^2}\,.$$

The equation of the line joining the centre of the osculating equilateral hyperbola with the given point on the curve is at once written down in its most general form, *viz.*, x, y being the coordinates of the point and X, Y the current coordinates, we have for the required equation

$$\frac{X - x}{Y - y} = \frac{x - \xi}{y - y} = \frac{r}{pr - 3q^2}\,,$$

which shews that the centre of the osculating equilateral hyperbola is on the axis of aberrancy, as is also geometrically evident. From the above values of ξ, η, it can be shown after some reductions that

$$\frac{d\xi}{dx} = \lambda_0\,T_0\,, \quad \frac{d\eta}{dx} = \mu_0\,T_0$$

where

$$\lambda_0 = \frac{9q^4 - r^2\,(1 + p^2)}{\left\{ r^2 + (rp - 3q^2)^2 \right\}^{\frac{1}{2}}}\,,$$

$$\mu_0 = \frac{r\,(1 + p^2)\,(6q^2 - pr) - 9pq^4}{\left\{ r^2 + (rp - 3q^2)^2 \right\}^2}\,,$$

$$T_0 = 9q^4 - 6pq^2r + (1 + p^2)\,(3qs - 4r^2)\,,$$

so that $T_0 = 0$ is the differential equation of all equilateral hyperbolas.

§ 3. *Geometric Interpretation.*

It is now extremely easy to give the true geometric interpretation of the differential equation of all parabolas; for we have shewn above that the index of aberrancy is given by the formula

$$I = \frac{3qs - 5r^2}{3q\left\{ r^2 + (rp - 3q^2)^2 \right\}^{\frac{1}{2}}}\,,$$

and the differential equation of all parabolas is

$$3qs - 5r^2 = 0.$$

Hence, we conclude that the required geometric interpretation is the property that *the index of aberrancy vanishes at every point of every parabola.*

§ 4. *Miscellaneous Theorems.*

The differential expression

$$3qs - 5r^2,$$

the vanishing of which we find to be the differential equation of all parabolas, may appropriately be taken to represent the species of the conic of closest contact at any point of a given curve. For, from the equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

we have

$$y = \mathrm{P}x + \mathrm{Q} \pm \sqrt{\mathrm{A}x^2 + 2\mathrm{H}x + \mathrm{B}},$$

where

$$\mathrm{P} = -\frac{h}{b}, \quad \mathrm{Q} = -\frac{f}{b},$$

$$\mathrm{A} = \frac{h^2 - ab}{b^2}, \quad \mathrm{H} = \frac{hf - bg}{b^2}, \quad \mathrm{B} = \frac{f^2 - bc}{b^2},$$

whence we have, as usual

$$\frac{d^2y}{dx^2} = q = \pm \frac{\mathrm{AB} - \mathrm{H}^2}{(\mathrm{A}x^2 + 2\mathrm{H}x + \mathrm{B})^{\frac{3}{2}}},$$

$$r = \mp \frac{3\,(\mathrm{AB} - \mathrm{H}^2)\,(\mathrm{A}x + \mathrm{H})}{(\mathrm{A}x^2 + 2\mathrm{H}x + \mathrm{B})^{\frac{5}{2}}},$$

$$s = \pm \frac{3\,(\mathrm{AB} - \mathrm{H}^2)\left\{4\,(\mathrm{A}x + \mathrm{H})^2 - (\mathrm{AB} - \mathrm{H}^2)\right\}}{(\mathrm{A}x^2 + 2\mathrm{H}x + \mathrm{B})^{\frac{7}{2}}};$$

Therefore, by actual calculation, we get

$$5r^2 - 3qs = \frac{9\mathrm{A}\,(\mathrm{AB} - \mathrm{H}^2)^2}{(\mathrm{A}x^2 + 2\mathrm{H}x + \mathrm{B})^4},$$

so that it is clear that the differential expression

$$5r^2 - 3qs$$

is of the same sign as

$$\mathrm{A} \text{ and } h^2 - ab.$$

Hence, we have the theorem that at any point of a curve, the conic of five-pointic-contact is an ellipse, hyperbola, or parabola, according as

$$5\left(\frac{d^3y}{dx^3}\right)^2 - 3\,\frac{d^2y}{dx^2}\,\frac{d^4y}{dx^4}$$

is negative, positive, or zero.*

Since we have proved that the radius of aberrancy is given by the formula

* See Dublin Examination Papers, 1875, p. 279, Ques. 4, by Prof. M. Roberts.

$$R = \frac{3\rho^2 \left\{ 9\rho^2 + \left(\frac{d\rho}{d\omega}\right)^2 \right\}}{9\rho^2 + 4\left(\frac{d\rho}{d\omega}\right)^2 - 9\rho \frac{d^2\rho}{d\omega^2}},$$

and as, moreover, in every parabola, the reciprocal of R vanishes, the differential equation of all parabolas in terms of ρ and ω is

$$3\rho \frac{d^2\rho}{d\omega^2} - 4\left(\frac{d\rho}{d\omega}\right)^2 - 9\rho^2 = 0.*$$

To integrate this, put

$$\rho = e^{\int u d\omega},$$

whence

$$3 \frac{du}{d\omega} = u^2 + 9,$$

or,

$$d\omega = \frac{3du}{u^2 + 9},$$

which gives

$$u = 3 \tan (\omega + k),$$

so that

$$\int u d\omega = 3 \int \tan (\omega + k)\, d\omega$$
$$= 3 \log m \sec (\omega + k),$$

and

$$\rho = e^{\int u dw} = m^3 \sec^3 (\omega + k),$$

which, therefore, is the relation between ρ and ω in every parabola, leading at once to the intrinsic equation

$$s = m^3 \int \sec^3 (\omega + k)\, d\omega,$$

and, if the origin be suitably chosen, we may put $k = 0$, so that we have the well-known result

$$s = m^3 \int \frac{d\omega}{\cos^3 \omega}.$$

14*th May*, 1888.

* See also P. A. S. B. 1888, p. 84, footnote.